arXiv:hep-th/0002161v2 7 Mar 2000

Quasi-Localization of Gravity by Resonant Modes

Csaba Csáki[a,*] **Joshua Erlich**[a] **and Timothy J. Hollowood**[a,b]

[a] *Theory Division T-8, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[b] *Department of Physics, University of Wales Swansea, Swansea, SA2 8PP, UK*
Email: `csaki@lanl.gov, erlich@lanl.gov, pyth@skye.lanl.gov`

ABSTRACT: We examine the behaviour of gravity in brane theories with extra dimensions in a non-factorizable background geometry. We find that for metrics which are asymptotically flat far from the brane there is a resonant graviton mode at zero energy. The presence of this resonance ensures quasi-localization of gravity, whereby at intermediate scales the gravitational laws on the brane are approximately four dimensional. However, for scales larger than the lifetime of the graviton resonance the five dimensional laws of gravity will be reproduced due to the decay of the four-dimensional graviton. We also give a simple classification of the possible types of effective gravity theories on the brane that can appear for general non-factorizable background geometries.

*J. Robert Oppenheimer Fellow.

The past two years have produced several surprising results in the field of gravity in extra dimensions. Firstly, Arkani-Hamed, Dimopoulos and Dvali [1] showed that the size of compact extra dimensions could be as large as a millimeter, with the fundamental Planck scale as low as a TeV, without running into contradiction with the current short-distance gravitational measurements, if the standard model fields are localized on a 4D "brane". Subsequently Randall and Sundrum (RS) [2] found that using a non-factorizable "warped" geometry for the extra dimension one could in fact have an infinitely large extra dimension, and still reproduce Newton's Law at large distances on the brane. The key observation of RS is that in their scenario there is a localized zero-energy graviton bound-state in 5D which should be interpreted as the ordinary 4D graviton. In this scenario the geometry of the extra dimension plays a crucial rôle and localization of gravity on the brane is impossible if the geometry far from the brane is asymptotically flat. In view of this fact, it is even more surprising that, as Gregory, Rubakov and Sibiryakov (GRS) [4] recently showed, even if the geometry is asymptotically flat far from the brane, it is still possible to find a phenomenologically viable model if one does not insist on the Newton potential being valid at arbitrarily large scales, but instead only requires it to hold over a range of intermediate scales (a related proposal can be found in [5]).

The aim of this paper is to present a physical explanation of the results of GRS and to provide a universal description of warped gravitational theories. We will show that the reason behind the GRS result is that even though there is no zero energy bound-state graviton in their model, there is a resonant mode—a "quasi bound-state"—at zero energy, which plays the rôle of the 4D graviton. The existence of this resonance at zero energy implies the "quasi-localization" of gravity—that is, it tends to produce a region of intermediate scales on which the gravitational laws appear to be four dimensional. We find that the long-distance scale at which gravity appears to be five dimensional again is inversely related to the width of the graviton resonance. The physics behind the appearance of this new scale is that at very large time scales the graviton decays into plane waves away from the brane, and thus reproduces 5D gravity at large distances. As the width of the resonance approaches zero, the lifetime becomes large and in the limit one regains the RS model. On the other hand, if the resonance becomes very wide, it is basically washed out from the spectrum, its effects become unimportant, and there is no longer a region where gravity is effectively 4D.

We show that the existence of such a resonance is expected in these kinds of theories when the geometry is asymptotically flat space far from the brane. Thus we find a simple way of classifying warped gravitational theories: if the ground state wave function is normalizable, one has localization of gravity à la Randall and Sundrum. If the ground state wavefunction is not normalizable, but the geometry does not asymptote to flat space, then there is simply no effective 4D gravity; however, if the ground state wavefunction is non-normalizable and the geometry asymptotes to flat space, we have quasi-localization of gravity via the resonance à la GRS.

The most general 5D metric with 4D Poincaré symmetry can be written

$$ds^2 \equiv g_{\mu\nu}dx^\mu \, dx^\nu = e^{-A(z)}\big(\eta_{ab}\, dx^a \, dx^b - dz^2\big) \ . \tag{1}$$

We will assume that the "warp factor" $A(z)$ is symmetric and, for simplicity, a non-decreasing function of z for $z > 0$. Furthermore we will also assume that the matter is localized on a brane at $z = 0$. We now consider fluctuations around the 4D Minkowski metric of the form $h_{ab}(x,z) = e^{3A(z)/4}\psi(z)\check{h}_{ab}(x)$, with $-\eta^{cd}\partial_c\partial_d\check{h}_{ab}(x) = m^2\check{h}_{ab}(x)$, where m is the four-dimensional Kaluza-Klein mass of the fluctuation. The behaviour of the fluctuation in the transverse space is governed by $\psi(z)$ which satisfies a Schrödinger-like equation:

$$-\frac{d^2\psi(z)}{dz^2} + V(z)\psi(z) = m^2\psi(z) \ , \qquad V(z) = \tfrac{9}{16}A'(z)^2 - \tfrac{3}{4}A''(z) \ . \tag{2}$$

Notice that (2) always admits a (not necessarily normalizable) zero-energy wavefunction

$$\hat{\psi}_0(z) = \exp\big[-\tfrac{3}{4}A(z)\big] \ , \tag{3}$$

which potentially describes the 4D graviton.

The auxiliary quantum system described by (2) encodes all the properties that we need in order to establish whether under favourable conditions there exists an effective 4D Newton potential on the brane. The relevant quantity to consider is the induced gravitational potential between two unit masses on the brane. A discrete (normalized) eigenfunction $\psi_m(z)$ of the quantum system contributes

$$\frac{\psi_m(0)^2}{M_*^3}\frac{e^{-mr}}{r} \ , \tag{4}$$

to the potential, where M_* is the fundamental Plank scale in 5D. On the other hand, continuum modes (normalized as plane waves asymptotically) contribute

$$\int_{m_0}^\infty dm \, \frac{\psi_m(0)^2}{M_*^3}\frac{e^{-mr}}{r} \ . \tag{5}$$

In the cases which we consider the potential $V(z)$ will always approach 0 as $|z| \to \infty$ and so $\hat{\psi}_0(z)$ is the only possible bound-state of the system and the continuum begins at $m_0 = 0$. The potential has the characteristic volcano shape with a central well surrounded by barriers that decay to zero [2, 3]. We can distinguish 3 classes depending on the asymptotic behaviour of $\hat{\psi}_0(z)$: (a) $\hat{\psi}_0(z)$ is normalizable and consequently falls off faster than $|z|^{-1/2}$; (b) $\hat{\psi}_0(z)$ is non-normalizable and falls off as a power $|z|^{-\alpha}$ ($\alpha \leq \tfrac{1}{2}$); and (c) $\hat{\psi}_0(z)$ is non-normalizable and asymptotes to a constant so the 5D spacetime is asymptotically flat. (Note that this case requires a region where $A''(z) < 0$, and therefore cannot arise as a scalar field domain wall in an otherwise flat background, for example.)
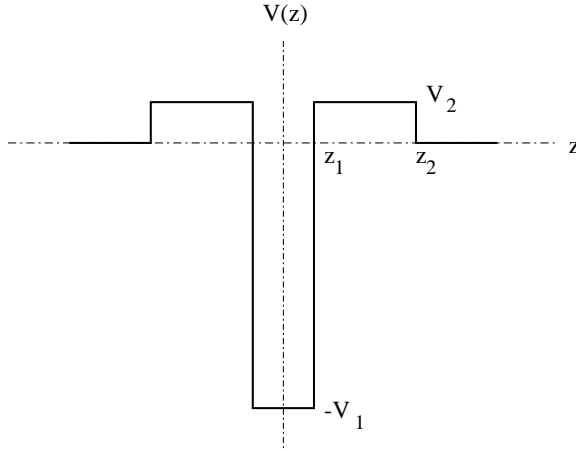
Figure 1: The volcano box potential.

For case (a), $\hat{\psi}_0(z)$ gives rise to the usual 4D Newton potential, as is apparent from (4). In [2,6] it was shown that the effects of the continuum modes generically give small corrections. The intuitive reason for this is that in order for $\hat{\psi}_0(z)$ to *be* normalizable, the tunneling probability for continuum modes of small m through the barriers of $V(z)$ must vanish as $m \to 0$; this implies that $\psi_m(0)^2$ must be less singular than m^{-1}. Hence the integral over the continuum modes in (5) always yields higher order corrections in r^{-1} relative to the leading r^{-1} piece.

For case (b) there are only continuum modes. In these cases $\psi_m(0)^2 \sim m^{-2\alpha}$, for small m, and so the potential (5) behaves as $r^{2\alpha-2}$ [6]. Consequently there is no effective 4D gravity. Note that in this case gravity does not necessarily appear 5D at large distances, either.

Case (c) is the main focus of interest. As in case (b) there is only a continuum contribution to the potential (5); however, we will argue that under favourable conditions an effective 4D Newton's Law is recovered as in [4] in some intermediate regime $r_1 \ll r \ll r_2$, where the parameters $r_{1,2}$ depend on details of the warp factor $A(z)$. This region of "quasi-localization" of gravity is intimately connected with the quasi-bound-state $\hat{\psi}_0(z)$ which causes a resonance in $\psi_m(0)^2$ at $m = 0$. The new long distance scale r_2 depends upon the width of the resonance: the narrower the resonance the larger the scale r_2. In the limit in which the width goes to zero $\hat{\psi}_0(z)$ becomes normalizable, $r_2 \to \infty$ and 4D gravity is obtained for all distance scales $\gg r_1$, as in the original RS model [2].

In order to prime our intuition it is useful to consider the "volcano box" potential (Figure 1), for which the continuum modes can be calculated exactly. The solution for a symmetric continuum

3

wavefunction has the form:

$$\psi_m(z) = \frac{1}{\sqrt{c(m)^2 + d(m)^2}} \begin{cases} \cos k_1 z & |z| \leq z_1 \\ a(m)\, e^{k_2(z-z_1)} + b(m)\, e^{-k_2(z-z_1)} & z_1 \leq |z| \leq z_2 \\ c(m)\, \cos k_3(z-z_2) + d(m)\, \sin k_3(z-z_2) & |z| \geq z_2 \ , \end{cases} \quad (6)$$

where

$$k_1 = \sqrt{V_1 + m^2} \ , \qquad k_2 = \sqrt{V_2 - m^2} \ , \qquad k_3 = \sqrt{m^2} \ , \quad (7)$$

and

$$\begin{aligned} c(m) &= \cos k_1 z_1 \cosh k_2(z_2 - z_1) - \tfrac{k_1}{k_2} \sin k_1 z_1 \sinh k_2(z_2 - z_1) \ , \\ d(m) &= \tfrac{k_2}{k_3} \cos k_1 z_1 \sinh k_2(z_2 - z_1) - \tfrac{k_1}{k_3} \sin k_1 z_1 \cosh k_2(z_2 - z_1) \ . \end{aligned} \quad (8)$$

In order to have a quasi-bound-state at $m = 0$ we require a solution of the form (6) with $\psi_0(z) = c(0)$ for $|z| \geq z_2$. This happens when the parameters of the potential satisfy

$$\sqrt{V_2} \tanh \sqrt{V_2}(z_2 - z_1) = \sqrt{V_1} \tan \sqrt{V_1} z_1 \ . \quad (9)$$

The quantity we are after is

$$\psi_m(0)^2 = \frac{1}{c(m)^2 + d(m)^2} \ . \quad (10)$$

For small m, we can expand $c(m)$ and $d(m)$ in powers of m. The important point is that $c(m)$ has an expansion in even powers of m while $d(m)$ has a expansion in odd powers of m:

$$c(m) = c_0 + c_1 m^2 + \cdots \ , \qquad d(m) = d_0 m + d_1 m^3 + \cdots \ . \quad (11)$$

Hence for small m, $\psi_m(0)^2$ has the Breit-Wigner form indicative of a resonance

$$\psi_m(0)^2 = \frac{\mathcal{A}}{m^2 + \Delta m^2} + \mathcal{O}(m^4) \ , \quad (12)$$

where the width of the resonance is $\Delta m = |c_0|/\sqrt{d_0^2 + 2c_0 c_1}$. The width depends in a complicated way on the parameters of the potential $V(z)$. A narrow resonance can be achieved by having c_0 small, which requires $\sqrt{V_2}(z_2 - z_1) \gg 1$. In this limit the width is approximately

$$\Delta m \simeq \frac{8}{(1 + V_2/V_1)(z_1 + 1/\sqrt{V_2})} e^{-2\sqrt{V_2}(z_2-z_1)} \ . \quad (13)$$

Intuitively, the behaviour of (13) can be deduced by the following reasoning: in order to get a narrow resonance we require that the tunneling probability for modes of small m through the barriers of the potential be very small, which is achieved by having $\sqrt{V_2}(z_2 - z_1) \gg 1$. In fact the

exponential factor in (13) can be deduced from a simple WKB analysis. In this approximation the tunneling probability for the eigenfunction $\psi_m(z)$ is

$$T(m) \sim \exp\left[-2\int_{z_1}^{z_2} dz \sqrt{V(z) - m^2}\right] = \exp\left[-2\sqrt{V_2 - m^2}(z_2 - z_1)\right] . \tag{14}$$

The width of the quasi-bound-state is $\Delta m \propto T(0)$, giving the exponential dependence in (13).

We expect the existence of a resonance to be generic, since it is caused by the non-normalizable mode $\hat{\psi}_0(z)$. Let us suppose that the resonance is sufficiently narrow that we can approximate $\psi_m(0)^2$ by

$$\psi_m(0)^2 = \frac{\mathcal{A}}{m^2 + \Delta m^2} + f(m) , \tag{15}$$

where $f(m)$ is some underlying function which rises from 0 to 1 as the energy goes from 0 to just over the height of the barrier. If we assume that $f(m) \sim m^\beta$ $(\beta > 0)$, for small m, the gravitational potential (5) is

$$U(r) = \frac{r_2 M_*^{-3} \mathcal{A}}{r} \int_0^\infty dx \frac{e^{-xr/r_2}}{x^2 + 1} + \mathcal{O}(1/r^{\beta+2}) , \tag{16}$$

where $r_2 = 1/\Delta m$. The contribution from the resonance gives the 4D Newton's Law for $r \ll r_2$ with Newton's constant

$$G_N = \frac{\pi r_2 \mathcal{A}}{2 M_*^3} , \tag{17}$$

whereas for $r \gg r_2$ the contribution goes as $1/r^2$ and so the 5D Newton potential is recovered at very large distances. The rise of the underlying part of the continuum $f(m)$ gives the short distance corrections to the 4D Newton's Law and sets the lower scale r_1.

The question now is under what conditions is the resonance narrow so that r_2 can be large. To investigate this we can, following our analysis of the volcano box potential, use the WKB approximation as a guide. The point is that the width of the resonance is proportional to the tunneling probability $T(m)$ for the continuum modes of zero energy, through the barrier of the potential, evaluated at $m = 0$. The WKB approximation gives

$$T(m) \sim \exp\left[-2\int_{z_1}^\infty dz \sqrt{V(z) - m^2}\right] , \tag{18}$$

where the integral is over the barrier region of the potential. Notice that the integral in (18) is convergent only when $V(z)$ falls off faster than z^{-2}; precisely the situation for case (c) above. In this case the limit $T(0)$ is finite and since we expect $\Delta m \propto T(0)$ the requirement for a narrow resonance is

$$\int_{z_1}^\infty dz \sqrt{V(z)} \gg 1 . \tag{19}$$

In other words, the barriers of the potential must be sufficiently powerful. For case (a) above, on the other hand, the integral diverges and $T(0) = 0$, as expected since $\hat{\psi}_0(z)$ is normalizable.

Now we consider some concrete examples. To recap what we need in order to get quasi-localized gravity on the brane is some geometry which is asymptotically flat in the transverse dimension. This was realized in the simple model of GRS [4] by patching together 5D AdS space onto 5D Minkowski space at some point $z = z_0$:

$$A(z) = \begin{cases} 2\log(k|z| + 1) & |z| \leq z_0 \\ 2\log(k|z_0| + 1) & |z| \geq z_0 \ . \end{cases} \tag{20}$$

This case can be exactly solved and one can show that quasi-localization occurs for $k^{-1} \ll r \ll k^2 z_0^3$. This allows us to check our simple WKB analysis; in this case for $z_0 \gg k^{-1}$ the WKB integral is $\int dz \sqrt{V(z)} \simeq \sqrt{15/4} \ln z_0$ giving $r_2 \propto z_0^{\sqrt{15}}$ (to compare with the exact behaviour $r_2 \propto z_0^3$). In the GRS scenario the patching of AdS to flat space requires the existence of new branes at z_0. However, this is not a necessary feature and one can invent scenarios which smoothly interpolate between the AdS geometry and flat space; for instance by taking

$$A(z) = -2\log\left(\frac{1}{k|z| + 1} + a\right) , \tag{21}$$

which can also be smoothed at $z = 0$ by taking, for example,

$$A(z) = -\log\left(\frac{1}{k^2 z^2 + 1} + a^2\right) . \tag{22}$$

In this case, the crossover occurs smoothly at $z_0 \simeq 1/(ka)$. For this example, it is a simple matter to numerically solve the differential equation (2) and find $\psi_m(0)^2$ as a function of m. Figure 2 illustrates this function for some values of the parameters giving rise to a fairly broad resonance so that the rise of $\psi_m(0)^2$ to 1 can also be seen. For small a the resonance becomes very narrow and one can verify numerically that it approximates the form in (15). Figure 3 shows a close-up of the resonance to illustrate the Breit-Wigner form. We can find the height of the resonance from our knowledge of $\hat{\psi}_0(z)$:

$$\frac{\mathcal{A}}{\Delta m^2} = \frac{2}{\pi}\hat{\psi}_0(0)^2 = e^{-3(A(0)-A(\infty))/2} \simeq \frac{2}{\pi}a^{-3} \ . \tag{23}$$

In addition, when the resonance is very narrow, we can find the width to leading order in a by using the fact that, as $a \to 0$, $\hat{\psi}_0(z)$ becomes normalizable and the effect of the resonance should appproximate a delta function. This gives

$$\frac{\mathcal{A}}{\Delta m} = \frac{4k}{\pi} \ . \tag{24}$$

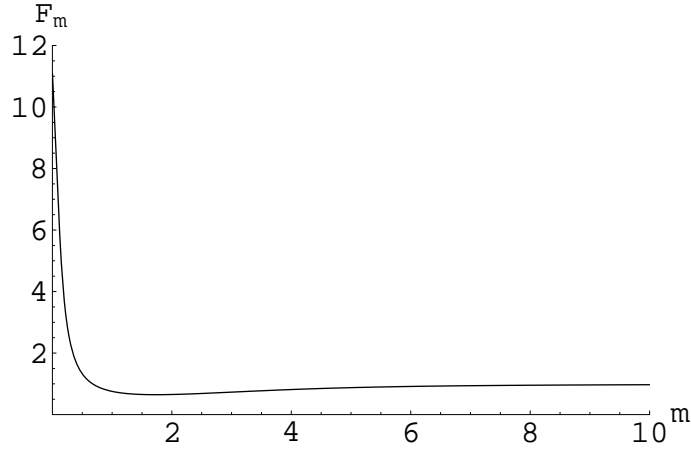Hence, $\Delta m \simeq 2ka^3 \simeq 2k^2 z_0^{-3} = r_2^{-1}$.

Figure 2: The quantity $F_m = \frac{\pi}{2}\psi_m(0)^2$ as a function of m for the geometry (22) (with $a = .25$, $k = 2$), which smoothly interpolates between AdS and Minkowski space. The resonance at $m = 0$ is clearly visible as well as the ultimate rise to 1 as the energy goes over the height of the barrier.
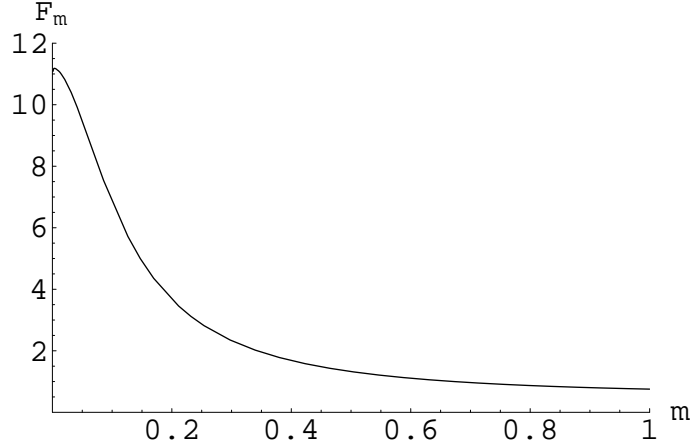

Figure 3: A close-up of the resonance in Figure 2.

We have introduced the idea of "quasi localization" of gravity on the brane as the general notion lying behind the scenario of GRS [4] whereby over a large range of intermediate distance scales the gravitational potential is, to a good accuracy, Newtonian. We argued that the relevant geometry for quasi localization is when the transverse geometry becomes asymptotically flat. The new large scale above which the gravitational potential becomes five-dimensional depends on the scale at which the crossover to flat space occurs. Physically we can describe the onset of 5D gravity on the brane at this new scale by saying that the effective 4D graviton is unstable and decays into the KK continuum. Obviously in order to be phenomenologically viable the lifetime must be very long. Although we have only considered the case with one extra dimension we expect that the same phenomenon to occur with any number of extra dimensions. In the same vein, we note that

quasi-localization may be of relevance in string theory where the transverse geometry to a p-brane soliton is indeed asymptotically flat.

Note added: After this paper has been completed a revised version of Ref. [4] (as well as Ref. [7]) appeared, which also independently noted the presence of the resonance in these theories.

References

[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429**, 263 (1998) [`hep-ph/9803315`]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436**, 257 (1998) [`hep-ph/9804398`].

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [`hep-th/9906064`].

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [`hep-ph/9905221`].

[4] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, `hep-th/0002072`.

[5] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, G. G. Ross and J. Santiago, `hep-ph/9912552`.

[6] C. Csáki, J. Erlich, T. J. Hollowood and Y. Shirman, `hep-th/0001033`.

[7] G. Dvali, G. Gabadadze and M. Porrati, `hep-th/0002190`.